

ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF FOURTH ORDER DIRICHLET PROBLEMS

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Abstract

The behaviour of solutions to fourth order problems is studied through the decomposition into a system of second order ones, which leads to relaxed formulations with the introduction of measure terms. This allows to solve a shape optimization problem for a simply supported thin plate.

1. Introduction

In this paper we study the asymptotic behaviour of solutions of fourth order elliptic problems on varying domains.

This has been widely studied in the past, in the case of second order elliptic operators (see for instance [6], [9], [7]). We will use such results decomposing fourth order differential equations into a system of second order ones.

Given a bounded open set U in \mathbb{R}^n , $n \geq 2$ and a function $f \in H^{-1}(\Omega)$, the fourth order equation

$$\begin{cases} \Delta^2 u = f \text{ in } H^{-1}(\Omega) \\ \Delta u \in H_0^1(\Omega) \\ u \in H_0^1(\Omega) \end{cases} \quad (1.1)$$

is linked to the model for the vertical displacement u of a thin plate, occupying a region U , simply supported on ∂U , subjected to a load f . Simply supported means that the boundary is fixed, but that the plate is free to rotate around the tangent to ∂U . For the general treatment of plate theory we refer to [15], [11], [4], [14], [5].

In particular we want to study the asymptotic behaviour of solutions when the domain varies. To this aim we will show (Proposition 3.1) that problem (1.1) is equivalent to the system of second order equations

$$\begin{cases} -\Delta u = v \text{ in } H^{-1}(U) \\ u \in H_0^1(U) \\ -\Delta v = f \text{ in } H^{-1}(U) \\ v \in H_0^1(U). \end{cases} \quad (1.2)$$

This problem can be handled with the theorems valid in the second order case (Theorems 2.1 and 2.2), and it will be proved that if u_n are the solutions of problems like (1.2) on a sequence of subdomains U_n of a given bounded domain Ω , then a subsequence of u_n converges weakly in $H_0^1(\Omega)$ to a function u_μ solving

$$\begin{cases} -\Delta u_\mu + \mu u_\mu = v \\ u_\mu \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \\ -\Delta v + \mu v = f \\ v \in H_0^1(\Omega) \cap L^2_\mu(\Omega). \end{cases} \quad (1.3)$$

where μ is a measure.

The study will be carried on for general fourth order elliptic operators with constant coefficients and no lower order terms, that can be splitted into two second order ones.

A motivation for the study of the asymptotic behaviour of solutions of Dirichlet problems in varying domains without geometric assumptions on the domains U_n are the so-called shape optimization problems: given a function $j : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ we consider the following problem

$$\min_{U \in \mathcal{U}(\Omega)} \int_{\Omega} j(x, u_U(x)) dx, \quad (1.4)$$

where $\mathcal{U}(\Omega)$ is the family of all open subset of Ω and u_U is the solution of the problem of type (1.1) in the set U .

In section 6 it will be shown that, in general, problem (1.4) does not have a solution. Hence a relaxed optimization problem will be introduced, where the set over which we minimize is the set of functions u_μ , where μ is a measure and u_μ solves the relaxed problem (1.3). This set is the closure of $\{u_U : U \in \mathcal{U}(\Omega)\}$ in $L^2(\Omega)$. This problem will always have solution and its minimum will coincide with the infimum of integral in (1.4).

This shape optimization problem, in the second order case, was studied in [1], [2], [3].

2. Notations and preliminary results

Given an open subset U of \mathbb{R}^n , $H_0^1(U)$ is the usual Sobolev Space, $H^{-1}(U)$ its dual, and $\langle \cdot, \cdot \rangle$ the duality pairing. If $U \subset \Omega$ and $u \in H_0^1(U)$, then the function

$$\tilde{u} := \begin{cases} u & \text{in } U \\ 0 & \text{in } \Omega \setminus U \end{cases}$$

is in $H_0^1(\Omega)$. From now on we will always denote, with the same symbol u , a function and its extension \tilde{u} .

In this paper we will deal with elliptic operators $L : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ of the form

$$Lu = \sum_{|\alpha|=m} c_\alpha \partial^\alpha u,$$

where α and β are multiindeces, and m is the order of the operator that will be 2 or 4. In any case they will be without lower order term and with constant coefficients.

The operators will be assumed to be elliptic in the sense that

$$\sum_{|\alpha|=m} c_\alpha \xi^\alpha \geq \gamma |\xi|^m, \quad \forall \xi \in \mathbb{R}^n,$$

where γ is a real positive constant. This, in our case, is the same as

$$P(\xi) \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad (2.1)$$

where P is the polynomial $\sum_{|\alpha|=m} c_\alpha \xi^\alpha$ associated to the operator L .

In this work, differential problems are always meant to be solved in the usual weak sense. This means, for instance, that, for $u \in H_0^1(U)$ the expression

$$-\Delta u = f \text{ in } H^{-1}(U)$$

is an equality of linear functionals

$$\langle -\Delta u, v \rangle = \sum_i \int_U \partial_i u \partial_i v \, dx = \langle f, v \rangle,$$

for any $v \in H_0^1(U)$.

As said above, the limit of a sequence of solutions of Dirichlet problems is not, in general, the solution of a problem of the same kind, but is the solution of a problem where a measure term appears. To deal with these problems we need to recall some notions.

For the notion of capacity of a set $E \subset \Omega$, which we will indicate by $\text{cap}(E)$, we refer to textbooks as [12] or [13]. We shall always identify a function $u \in H_0^1(\Omega)$ with its quasi-continuous representative.

Now, let $\mathcal{M}_0(\Omega)$ be the set of Borel measures which are zero on the sets of zero capacity.

For one such measure $\mu \in \mathcal{M}_0(\Omega)$, $L_\mu^2(U)$ will be the space of functions such that

$$\int_U |u|^2 \, d\mu < +\infty.$$

Given a second order operator $A = \sum_{ij} a_{ij} \partial_i \partial_j$, with constant coefficients, for a function $u \in H_0^1(U) \cap L_\mu^2(U)$ to solve the equation

$$Au + \mu u = f,$$

will mean that

$$\sum_{i=1}^n \int_U a_{ij} \partial_j u \partial_i v \, dx + \int_U uv \, d\mu = \int_U fv \, dx, \quad (2.2)$$

for all test functions in $v \in H_0^1(U) \cap L_\mu^2(U)$.

It can be easily proved that the space $H_0^1(U) \cap L_\mu^2(U)$ is a Hilbert space whenever μ is in $\mathcal{M}_0(\Omega)$, and hence, by Lax-Milgram Lemma, we have existence and uniqueness of solutions for a problem of the form

$$\begin{cases} Au + \mu u = f \\ u \in H_0^1(U) \cap L_\mu^2(U), \end{cases}$$

for any linear elliptic second order operator A .

The decomposition of a fourth order problem in a system of two second order equations allows us to study the asymptotic behaviour applying well known theorems for the second order case separately to each equation. The following results, that can be found, for instance in [9], [7], are the key points of the theory.

Theorem 2.1. *Let A be a second order elliptic operator, as described above. For every sequence $\{\mu_n\}$ in $\mathcal{M}_0(\Omega)$ there exists a subsequence μ_{n_k} such that, for every sequence $\{g_n\}$ in $H^{-1}(\Omega)$, strongly convergent to $g \in H^{-1}(\Omega)$, we have*

$$z_{n_k} \rightharpoonup z \text{ weakly in } H_0^1(\Omega),$$

where z_{n_k} and z solve

$$\begin{cases} Az_{n_k} + \mu_{n_k} z_{n_k} = g_{n_k} \\ z_{n_k} \in H_0^1(\Omega) \cap L_{\mu_{n_k}}^2(\Omega), \end{cases} \quad \begin{cases} Az + \mu z = g \\ u \in H_0^1(\Omega) \cap L_\mu^2(\Omega), \end{cases}$$

respectively.

Theorem 2.2. *For any measure $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence U_n of open subsets of Ω such that, for any sequence $\{g_n\}$ in $H^{-1}(\Omega)$, strongly convergent to $g \in H^{-1}(\Omega)$, the solutions z_n of the problems*

$$\begin{cases} Az_n = g_n & \text{in } H^{-1}(U_n) \\ z_n \in H_0^1(U_n) \end{cases}$$

converge weakly in $H_0^1(\Omega)$ to the solution z of

$$\begin{cases} Az + \mu z = g \\ u \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

3. Decomposition of fourth order operators

Fourth order elliptic problems are studied mainly with two different kinds of boundary conditions. In the model case of the bi-laplacian, they correspond to two different physical problems regarding, as said above, the displacement of a thin plate. Problems of the type

$$\begin{cases} \Delta^2 u = f \text{ in } H^{-1}(\Omega) \\ u \in H_0^2(\Omega) \end{cases}$$

correspond to having $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, that is to say that the plate is clamped along its boundary. The asymptotic behaviour of the solutions of such problems has been studied in [10].

In this work we deal with the second kind of boundary conditions, as in (1.1), and we do it decomposing the problem in a system of two second order equations.

The decomposability of the fourth order operator

$$Lu = \sum_{|\alpha|=4} c_\alpha \partial^\alpha u$$

can be seen through the associated polynomial

$$P(\xi) = \sum_{|\alpha|=4} c_\alpha \xi^\alpha.$$

It is a simple algebraic fact that, if the polynomial can be splitted into two second degree polynomials

$$P(\xi) = Q(\xi)R(\xi),$$

then other decompositions can be obtained only by exchanging the order or multiplying and dividing by constants. Observe that, according to (2.1), if P is elliptic, then so are R and Q .

We remark here that such a decomposition can always be done in the two dimensional case. In higher dimensions this is not always possible.

So assume that

$$Lu = \sum_{|\alpha|=4} c_\alpha \partial^\alpha = \sum_{i,j=1}^n b_{ij} \partial_j \partial_i \left(\sum_{k,l=1}^n a_{kl} \partial_k \partial_l \right) u = BAu.$$

Proposition 3.1. *Let U be a subset of Ω , and $A = \sum_{i,j=1}^n a_{ij} \partial_j \partial_i$, $B = \sum_{i,j} b_{ij} \partial_j \partial_i$ be second order elliptic operators with constant coefficients. Let $f \in H^{-1}(U)$. The following three problems are equivalent:*

$$(i) \begin{cases} BAu = f \text{ in } H^{-1}(U) \\ Au \in H_0^1(U) \\ u \in H_0^1(U) \end{cases} \quad (ii) \begin{cases} u \in H_0^1(U) : Au \in L^2(U) \\ \int_U Au B \varphi dx = \langle f, \varphi \rangle \\ \forall \varphi \in H_0^1(U) : B\varphi \in L^2(U) \end{cases} \quad (iii) \begin{cases} Au = v \text{ in } H^{-1}(U) \\ u \in H_0^1(U) \\ Bv = f \text{ in } H^{-1}(U) \\ v \in H_0^1(U) \end{cases} \quad (3.1)$$

Proof. $(i) \Rightarrow (ii)$. The equality $\langle BAu, \varphi \rangle = \langle f, \varphi \rangle$ holds in particular for any φ in $H_0^1(U)$ such that $B\varphi \in L^2(U)$. Since B is symmetric and $Au \in H_0^1(U)$ we get

$$\int_U Au B \varphi dx = \langle B\varphi, Au \rangle = \langle BAu, \varphi \rangle = \langle f, \varphi \rangle$$

$(ii) \Rightarrow (iii)$. Let v be solution of

$$v \in H_0^1(U), \quad \langle Bv, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in H_0^1(U).$$

Then

$$\langle B\varphi, v \rangle = \langle f, \varphi \rangle$$

and if, in particular, φ is such that $B\varphi \in L^2(U)$, we get

$$\int_U v B \varphi dx = \langle f, \varphi \rangle,$$

which, subtracted to the equation in (ii) gives

$$\int_U (Au - v) B \varphi dx = 0.$$

Observe now that, thanks to Lax-Milgram theorem, every function in $L^2(U)$ can be written as $B\varphi$, with $\varphi \in H_0^1(U)$, so we obtain

$$\int_U (Au - v) z dx = 0, \quad \forall z \in L^2(U).$$

Taking $z = Au - v$, we get that $\|Au - v\|_{L^2(U)}^2 = 0$, hence $Au = v$, and u solves problem (iii).

(iii) \Rightarrow (ii). If u is a solution of (iii) then

$$\langle Bv, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in H_0^1(U).$$

and if, in particular, $B\varphi \in L^2(U)$ we have

$$\int_U v B\varphi \, dx = \langle f, \varphi \rangle.$$

Since now $Au = v$ we have

$$\int_U Au B\varphi \, dx = \langle f, \varphi \rangle,$$

for every $\varphi \in H_0^1(U)$ with $B\varphi \in L^2(U)$, hence u solves (ii).

(ii) \Rightarrow (i). We have already proved that from (ii) it follows that $Au \in H_0^1(U)$, hence $BAu \in H^{-1}(U)$, and so

$$\langle f, \varphi \rangle = \int_U Au B\varphi \, dx = \langle B\varphi, Au \rangle = \langle BAu, \varphi \rangle;$$

that is, $BAu = f$ in $H^{-1}(U)$

□

Remark 3.2. The equivalence of these problems gives for free existence and uniqueness, since this is true for problem (iii) thanks to the observation made in section 2. This is not obvious for problem (ii) because the spaces of solution and of test functions are different.

Remark 3.3. Notice that if the boundary of U is regular, then by regularity theorems, $Au \in L^2(U)$ implies that u belongs to $H^2(U)$. The same is true for the test functions φ . So the problem(3.1)(ii) becomes:

$$\begin{cases} u \in H_0^1(U) \cap H^2(U) \\ \int_U Au B\varphi \, dx = \langle f, \varphi \rangle \\ \forall \varphi \in H_0^1(U) \cap H^2(U). \end{cases}$$

Remark 3.4. It is important to remark that the boundary conditions in the equivalent problems (3.1)(i)(ii)(iii) depend on the choice of the decomposition $L = BA$.

If we have two decompositions, in the sense that

$$L\varphi = B_1 A_1 \varphi = B_2 A_2 \varphi, \quad \forall \varphi \in \mathcal{D}'(\Omega),$$

then the boundary conditions in (3.1)(i) are different. Hence also the other problems (ii) and (iii) differ. Had we chosen the Dirichlet boundary conditions, that is seeking $u \in H_0^2(U)$, all decompositions would have given the same solution. But in this case what makes the difference is the boundary condition, as can be seen with the following example.

Example 3.5. The operator

$$Lu := u_{xxxx} + 2u_{yyyy} + 3u_{xxyy}$$

can be written, for instance, as the product of

$$Au = \Delta u \quad \text{and} \quad Bu = \Delta u + u_{yy}.$$

But the integral in equation (3.1)(ii) will be different according to which operator we will apply first:

$$\begin{aligned} \int AuBv \, dx &= \int (u_{xx}v_{xx} + 2u_{yy}v_{yy} + 2u_{xx}v_{yy} + u_{yy}v_{xx}) \, dx, \\ \int BuAv \, dx &= \int (u_{xx}v_{xx} + 2u_{yy}v_{yy} + u_{xx}v_{yy} + 2u_{yy}v_{xx}) \, dx. \end{aligned}$$

Computations show that these two integrals differ by a term on the boundary, which would vanish if functions were in $H_0^2(U)$.

4. The asymptotic behaviour

We come now to examine problem (3.1)(i) when we have a sequence of domains U_n all contained in Ω . Let f be in $H^{-1}(\Omega)$ (this implies it is also in $H^{-1}(U_n)$ for any U_n) and consider always functions of $H_0^1(U_n)$ trivially extended to the whole of Ω . As proved in Proposition 3.1, we can study directly

$$\begin{cases} Au_n = v_n & \text{in } H^{-1}(U_n) \\ u_n \in H_0^1(U_n) \\ Bv_n = f & \text{in } H^{-1}(U_n) \\ v_n \in H_0^1(U_n). \end{cases} \quad (4.1)$$

We first consider the second equation

$$\begin{cases} Bv_n = f \text{ in } H^{-1}(U_n) \\ v_n \in H_0^1(U_n). \end{cases}$$

From Theorem 2.1 we know that there exist a subsequence, which we still call U_n , and a measure μ_B , depending on the sequence U_n , on the operator B , but not on f , such that

$$v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)$$

and v solves

$$\begin{cases} Bv + \mu_B v = f \\ v \in H_0^1(\Omega) \cap L_{\mu_B}^2(\Omega), \end{cases} \quad (4.2)$$

in the sense specified in (2.2).

We can now apply Theorem 2.1 to the problem in u

$$\begin{cases} Au_n = v_n \text{ in } H^{-1}(U_n) \\ u_n \in H_0^1(U_n), \end{cases}$$

taking as U_n only those in the subsequence obtained for B . Again there exists a subsequence, which we still call U_n , and a measure μ_A , depending on the sequence U_n , on the operator A , but not on the sequence v_n , such that

$$u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega)$$

and u solves

$$\begin{cases} Au + \mu_A u = v \\ u \in H_0^1(\Omega) \cap L_{\mu_A}^2(\Omega). \end{cases} \quad (4.3)$$

This allows us to conclude that, if u_n are the solutions of system (4.1) then, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega)$$

and u is the solution of

$$\begin{cases} Au + \mu_A u = v \\ u \in H_0^1(\Omega) \cap L_{\mu_A}^2(\Omega) \\ Bv + \mu_B v = f \\ v \in H_0^1(\Omega) \cap L_{\mu_B}^2(\Omega). \end{cases} \quad (4.4)$$

5. The single equation formulation

The goal of this section is to write problem (4.4) as a single equation of the form

$$\int_{\Omega} (Au + \mu_A u)(B\varphi + \mu_B \varphi) dx = \langle f, \varphi \rangle.$$

Of course $Au + \mu_A u$ alone doesn't make sense, because it has to be understood in the sense explained in chapter 2. What we will do now, is hence to define suitable function spaces, which will play the role of $\{z \in H_0^1(\Omega) : Az \in L^2(\Omega)\}$ and $\{w \in H_0^1(\Omega) : Bw \in L^2(\Omega)\}$ and operators Λ_{μ_A} and Λ_{μ_B} which give meaning to the integral

$$\int_{\Omega} \Lambda_{\mu_A} u \Lambda_{\mu_B} \varphi dx.$$

The construction of such operators will be done only relatively to operator A , being the one relative to B perfectly analogous.

Consider the operator

$$\begin{aligned} R_A : H^{-1}(\Omega) &\longrightarrow H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega) \\ z &\longmapsto w \end{aligned}$$

where w is the solution of

$$\begin{cases} Aw + \mu_A w = z \\ w \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega); \end{cases} \quad (5.1)$$

so that we have $u = R_A v$ (and $v = R_B f$).

Define now the space

$$V_A(\Omega) := \{w \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega) : \exists z \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega) \text{ s.t. } w = R_A z\}.$$

Let us prove that the function z is unique, for each w .

Assume, by contradiction, that there exist two $z_1, z_2 \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega)$ such that $w = R_A z_1 = R_A z_2$. From the equation we have

$$\int_{\Omega} (z_1 - z_2) \psi dx = 0, \quad \forall \psi \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega).$$

Since $z_1 - z_2 \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega)$ we get

$$\int_{\Omega} (z_1 - z_2)^2 dx = \|z_1 - z_2\|_{L^2(\Omega)}^2 = 0$$

and we conclude that $z_1 = z_2$.

So we have defined

$$V_A(\Omega) = R_A \left(H_0^1(\Omega) \cap L_{\mu_A}^2(\Omega) \right)$$

and on its image R_A is one-to-one. So, on the space $V_A(\Omega)$, the inverse operator can be defined:

$$\Lambda_{\mu_C} : V_A(\Omega) \longrightarrow H_0^1(\Omega) \cap L_{\mu_A}^2(\Omega).$$

$$w \quad \longmapsto \quad z \text{ s.t. } w = R_A z$$

Observe now that the operator R_A is symmetric, that is:

$$\langle h, R_A g \rangle = \langle g, R_A h \rangle,$$

for any $h \in H^{-1}(\Omega)$ and $g \in H_0^1(\Omega)$.

We are now able to write system (4.4) as a single equation. We have $v = R_B f$ and $u = R_A v$, so that $u \in V_A(\Omega)$. For any $\varphi \in V_B(\Omega)$ there exists a unique $z \in H_0^1(\Omega) \cap L_{\mu_B}^2(\Omega)$ such that $\varphi = R_B z$. Hence,

$$\begin{aligned} \langle f, \varphi \rangle &= \langle f, R_B z \rangle = \langle z, R_B f \rangle = \langle z, v \rangle \\ &= \langle \Lambda_{\mu_B} \varphi, \Lambda_{\mu_A} u \rangle = \int_{\Omega} \Lambda_{\mu_B} \varphi \Lambda_{\mu_A} u \, dx. \end{aligned}$$

Last equality holds because both $\Lambda_{\mu_B} \varphi$ and $\Lambda_{\mu_A} u$ are in $L^2(\Omega)$.

Hence we can conclude that the function u , solution of (4.4), also solves

$$\begin{cases} u \in V_A(\Omega) \\ \int_{\Omega} \Lambda_{\mu_B} \varphi \Lambda_{\mu_A} u \, dx = \langle f, \varphi \rangle \\ \forall \varphi \in V_B(\Omega). \end{cases}$$

It is easily seen that also the converse holds. Let v be a solution of

$$\begin{cases} Bv + \mu_B v = f \\ v \in H_0^1(\Omega) \cap L_{\mu_B}^2(\Omega); \end{cases}$$

then, by the definition, $v = R_B f$. Hence, taken any $z \in H_0^1(\Omega) \cap L_{\mu_B}^2(\Omega)$, we set $\varphi := R_B z$, and we have:

$$\begin{aligned} \langle v, z \rangle &= \langle R_B f, z \rangle = \langle f, R_B z \rangle = \langle f, \varphi \rangle \\ &= \langle \Lambda_{\mu_A} u, \Lambda_{\mu_B} \varphi \rangle = \langle \Lambda_{\mu_A} u, z \rangle. \end{aligned}$$

Taking then $z = v - \Lambda_{\mu_A} u$, we obtain

$$\|v - \Lambda_{\mu_A} u\|_{L^2}^2 = 0,$$

so

$$\Lambda_{\mu_A} u = v,$$

that is, by the definition of Λ_{μ_A} ,

$$u = R_A v,$$

hence u solves (4.4).

All this can be summarized in the following

Theorem 5.1. *A function $u \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega)$ is a solution of*

$$\begin{cases} u \in V_A(\Omega) \\ \int_{\Omega} \Lambda_{\mu_B} \varphi \Lambda_{\mu_A} u \, dx = \langle f, \varphi \rangle \\ \forall \varphi \in V_B(\Omega), \end{cases}$$

if and only if it solves

$$\begin{cases} Au + \mu_A u = v \\ u \in H_0^1(\Omega) \cap L^2_{\mu_A}(\Omega) \\ Bv + \mu_B v = f \\ v \in H_0^1(\Omega) \cap L^2_{\mu_B}(\Omega). \end{cases}$$

It is important to remark that the road back to the equivalence with a fourth order problem stops here. This is because, in general, the image of Λ_{μ_A} is not contained in the domain of Λ_{μ_B} , even if $A = B$ and the two measures coincide.

6. The optimization problem

Let us consider now the optimization problem (1.4).

Let f be a function in $H^{-1}(\Omega)$ and let $j : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the standard Carathéodory conditions and be such that

$$|j(x, s)| \leq b(x) + \beta |s|^p, \quad \text{for a.e. } x \in \Omega \text{ and } \forall s \in \mathbb{R}, \quad (6.1)$$

with $b \in L^1(\Omega)$, $\beta \in \mathbb{R}$ and $1 \leq p < 2^*$, where $2^* = \frac{2n}{n-2}$ is the Sobolev exponent.

We want to study

$$\min_{U \in \mathcal{U}(\Omega)} \int_{\Omega} j(x, u_U(x)) dx, \quad (6.2)$$

where $\mathcal{U}(\Omega)$ is the family of all open subsets of Ω and u_U is the solution, trivially extended in $\Omega \setminus U$, of the problem

$$\begin{cases} \Delta^2 u = f & \text{in } H^{-1}(U) \\ \Delta u \in H_0^1(U) \\ u \in H_0^1(U). \end{cases}$$

In general problem (6.2) does not have solution, in the sense that the infimum is not attained on the set $\{u_U : U \in \mathcal{U}(\Omega)\}$. This can be seen with the following example.

Example 6.1. Consider the case of $\Omega \subset \mathbb{R}^n$ with smooth boundary. Consider a function

$$w \in C^\infty(\bar{\Omega}), \text{ s.t. } w > 0 \text{ in } \Omega, \quad w = 0 \text{ and } \Delta w = 0 \text{ on } \partial\Omega$$

and set

$$z := -\Delta w + w, \quad f := -\Delta z + z. \quad (6.3)$$

Consider now

$$j(x, s) = (s - w(x))^2,$$

so that the integral

$$\int_{\Omega} (u(x) - w(x))^2 dx \quad (6.4)$$

takes the value zero if and only if $u = w$ a.e. It will turn out from the theory (see equation (6.7)) that zero is in fact the infimum of the functional over all solutions of Dirichlet problems on subsets of Ω .

We prove now that, with this choice of f , w can not be the solution of the problem

$$\begin{cases} \Delta^2 u_U = f & \text{in } H^{-1}(U) \\ \Delta u_U \in H_0^1(U) \\ u_U \in H_0^1(U), \end{cases}$$

for any $U \subset \Omega$, or, equivalently (thanks to Proposition 3.1), of system

$$\begin{cases} -\Delta u_U = v & \text{in } H^{-1}(U) \\ u_U \in H_0^1(U) \\ -\Delta v = f & \text{in } H^{-1}(U) \\ v \in H_0^1(U). \end{cases}$$

First consider the case when $\text{cap}(\Omega \setminus U) = 0$. Then the spaces $H_0^1(\Omega)$ and $H_0^1(U)$ coincide (see for instance Theorem 4.5 in [13]), and the problem in U is the same as the problem in Ω . We show that $w \neq u_\Omega$. From (6.3) we have

$$\begin{aligned}\Delta^2 w - \Delta w &= -\Delta z \\ &= f - z \\ &= f + \Delta w - w.\end{aligned}$$

So $\Delta^2 w = f + 2\Delta w - w$, and hence w can be such that $\Delta^2 w = f$ only if $-\Delta w = -\frac{w}{2}$ but this is impossible because, by the maximum principle, w should be negative, while we have chosen it strictly positive.

The second case is when $\text{cap}(\Omega \setminus U) > 0$. If this happens u_U has to be zero in $\Omega \setminus U$, so that $u_U \neq w$ on a set of non-zero capacity. But for functions in $H_0^1(\Omega)$ to be equal Lebesgue-a.e. is the same as cap-a.e., hence u_U can not be a minimizer for (6.4).

The example shows then, that to be able to solve always our optimization problem, it is convenient to seek our minima in the larger set $M := \{u_\mu : \mu \in \mathcal{M}_0(\Omega)\}$, which is the closure of $N := \{u_U : U \in \mathcal{U}(\Omega)\}$ in the weak topology of $H_0^1(\Omega)$. In other words we introduce the relaxed optimization problem

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_{\Omega} j(x, u_\mu(x)) dx, \quad (6.5)$$

where u_μ denotes the solution to the relaxed problem

$$\begin{cases} -\Delta u_\mu + u_\mu \mu = v_\mu \\ u_\mu \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \\ -\Delta v_\mu + v_\mu \mu = f \\ v_\mu \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \end{cases} \quad (6.6)$$

and each equation is meant in the sense of formula (2.2).

In order to see that solving the new problem (6.5) gives the complete solution to problem (6.2), we have to show two things: that (6.5) has a solution and that

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_{\Omega} j(x, u_\mu(x)) dx = \inf_{U \in \mathcal{U}(\Omega)} \int_{\Omega} j(x, u_U(x)) dx. \quad (6.7)$$

As for the first thing, we need to show that the set M is closed in the weak topology of $H_0^1(\Omega)$. So consider minimizing sequence μ_n and let u_{μ_n} be the solutions of the

systems

$$\begin{cases} -\Delta u_{\mu_n} + u_{\mu_n} \mu_n = v_{\mu_n} \\ u_{\mu_n} \in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega) \\ -\Delta v_{\mu_n} + v_{\mu_n} \mu_n = f \\ v_{\mu_n} \in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega). \end{cases}$$

Applying Theorem 2.1 to the second equation we have that there exist a subsequence, called for simplicity μ_n , and a measure $\mu \in \mathcal{M}_0(\Omega)$ such that $v_{\mu_n} \rightharpoonup v_\mu$ weakly in $H_0^1(\Omega)$,

$$\begin{cases} -\Delta v_\mu + v_\mu \mu = f \\ v_\mu \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

Applying again Theorem 2.1 to the first equation (only those corresponding to the subsequence), we obtain that

$$u_{\mu_n} \rightharpoonup u_\mu \text{ weakly in } H_0^1(\Omega).$$

and u_μ is the solution of (6.6).

Hence u_μ is admissible and, by the minimizing property of the sequence $\{\mu_n\}$,

$$\int_{\Omega} j(x, u_\mu(x)) dx = \min_{\nu \in \mathcal{M}_0(\Omega)} \int_{\Omega} j(x, u_\nu(x)) dx.$$

On the other hand observe that every problem of the form

$$\begin{cases} -\Delta z = w \\ z \in H_0^1(U) \\ -\Delta w = g \\ w \in H_0^1(U) \end{cases}$$

can be viewed as a relaxed problem with the measure

$$\mu^U(B) := \begin{cases} 0 & \text{if } B \setminus U \text{ has capacity zero} \\ +\infty & \text{otherwise} \end{cases}$$

since this holds for second order problems as can be seen, for instance, in [2].

This ensures us that problem (6.5) is indeed an extension of (6.2), in the sense that $N \subseteq M$, so that

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_{\Omega} j(x, u_\mu(x)) dx \leq \inf_{U \in \mathcal{U}(\Omega)} \int_{\Omega} j(x, u_U(x)) dx.$$

To prove the inverse inequality we need the following

Lemma 6.2. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then there exists a sequence U_n of subsets of Ω such that*

$$u_{U_n} \rightharpoonup u_\mu \text{ weakly in } H_0^1(\Omega)$$

where u_{U_n} and u_μ solve respectively

$$\begin{cases} -\Delta u_{U_n} = v_n & \text{in } H^{-1}(U_n) \\ u_{U_n} \in H_0^1(U_n) \\ -\Delta v_n = f & \text{in } H^{-1}(U_n) \\ v_n \in H_0^1(U_n) \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_\mu + u_\mu \mu = v_\mu \\ u_\mu \in H_0^1(\Omega) \cap L_\mu^2(\Omega) \\ -\Delta v_\mu + v_\mu \mu = f \\ v_\mu \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

Proof. We know, from Theorem 2.2, that given μ , we have a sequence of sets U_n such that, for every $g \in H^{-1}(\Omega)$, if

$$\begin{cases} -\Delta z_n = g & \text{in } H^{-1}(U_n) \\ z_n \in H_0^1(U_n). \end{cases}$$

then the sequence z_n tends weakly in $H_0^1(\Omega)$ to z_μ , solution of

$$\begin{cases} -\Delta z_\mu + v_\mu \mu = g \\ z_\mu \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

Since the sequence U_n depends only on the operator and not on the right hand side the same sequence of sets suits for the problem in u . Hence u_{U_n} converge to u_μ solution of the second system. \square

Since, by the hypothesis (6.1), the operator $u \mapsto \int j(x, u) dx$ is continuous in the strong topology of $L^2(\Omega)$ (see [2] Theorem 4.1), the density result enables us to conclude that

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_{\Omega} j(x, u_\mu(x)) dx = \inf_{U \in \mathcal{U}(\Omega)} \int_{\Omega} j(x, u_A(x)) dx.$$

This ensures us also that every solution of a relaxed system can be approximated by solutions of fourth order problems.

Remark 6.3. We remark here that the model case of the bi-laplacian can be extended without changes in the proofs to the case of a fourth order elliptic operator with constant coefficients L such that

$$Lu = C^2 u,$$

where C is a second order elliptic operator with constant coefficients.

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